

# Stochastic Stability Analysis and Synthesis of Continuous-Time Linear Networked Systems

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## Abstract

In this paper, we study the problem of stability analysis and controller synthesis of continuous-time linear networked systems in the presence of stochastic uncertainty. Stochastic uncertainty is assumed to enter multiplicatively in system dynamics through input and output channels of the plant. We used the mean square notion for stochastic stability to address the analysis and controller synthesis problems of linear networked systems. These results generalize existing results on stability analysis and controller synthesis from discrete-time linear systems to continuous-time linear systems with multiplicative uncertainty. Necessary, sufficient conditions for mean square exponential stability are expressed in terms of the input-output property of deterministic or nominal system dynamics captured by the *mean square* system norm and variance of channel uncertainty. The stability results can also be interpreted as small gain theorem for continuous-time stochastic systems. Linear Matrix Inequalities (LMI)-based optimization formulation is provided for the computation of mean square system norm for stability analysis and controller synthesis. For a special case of single input channel uncertainty, we also prove a fundamental limitation result that arise in the mean square exponential stabilization of continuous-time linear systems. Simulation results are presented for WSCC 9 bus power system to demonstrate the application of developed framework.

## I. INTRODUCTION

The problem of stability analysis and control synthesis of systems in the presence of uncertainty has a rich, long history of literature. The literature in this area can be broadly divided into two parts. Classical robust control literature addresses this problem using norm bounds on uncertainty [1], [2]. In this paper, we study the robust control problem for continuous-time linear dynamics, where the uncertainty is modeled as a stochastic random variable. The stochastic uncertainty is assumed parametric and hence enters multiplicatively in system dynamics. The analysis and control problem with stochastic multiplicative uncertainty have received renewed attention lately as a model for network controlled system with communication uncertainty.

Some of the classical results involving stochastic stability analysis and control problems are presented in [3]. The work by Wonham [4] is one of the earliest literature on this topic involving continuous-time dynamics with multiplicative measurement and control noise. In [5], frequency domain-based stability criteria for continuous-time LTI system with state dependent noise is derived. The authors in [6] study the LQR problem for continuous-time linear systems with state-dependent noise entering only in the state dynamics. In [7], mean square exponential stability analysis and static state feedback control design for stochastic systems with state-dependent control noise are studied. The same authors, using state feedback control, developed robust stabilization results for continuous and discrete-time uncertain LTI systems in [8].

In [9], [10], using state feedback, the authors propose an input-output operator approach for characterizing the stability radii and maximizing the stability radii. In [11], the author provides a comparison of necessary and sufficient conditions with

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dynamic and static output feedback controller involving stochastic multiplicative uncertainty and deterministic norm bounded uncertainty respectively. Bernstein [11] also provide comprehensive survey of literature on this topic of stochastic stability analysis and control. In [12], a linear matrix inequality (LMI)-based mean square exponential stability result using static state feedback control is given for continuous-time LTI systems with state dependent noise. Using input/output operator approach, a small-gain theorem for stochastic systems with state-dependent noise only affecting the state dynamics has been developed in [13]. In contrast to these references, we develop mean square exponential stability analysis and synthesis results with stochastic multiplicative uncertainty, both at the input and output side of the plant. The problem formulation is general enough to address problems involving not only input-output channel uncertainty but also parametric stochastic uncertainty.

There is also extensive literature on this topic for systems involving nonlinear dynamics with multiplicative stochastic uncertainty. In [14], [15], the stability analysis and stabilization of nonlinear systems for stochastic uncertainty of unknown covariance is studied generalizing results from [4], [5], [12] to nonlinear dynamics. Lyapunov function-based approach for stability analysis and stabilization for nonlinear stochastic system is proposed in [16]. Generalization of stochastic positive real lemma for stochastic stability analysis and stabilization of nonlinear system in Lure form are addressed in [17]. The references in the above mentioned papers further provide more literature review on stochastic system stability analysis.

Research activities in the area of network controlled system have lead to the renewed interest in the analysis and design of systems with multiplicative uncertainty [18]. In particular, network systems with erasure or time-delay uncertainty in the input or output communication channel can be modeled as a system with multiplicative uncertainty. Issues related to fundamental limitations for stabilization and estimation of networked systems, i.e., largest tolerable channel uncertainty are addressed in [18]–[23]. Fundamental limitation results are extended to nonlinear systems in [24], [25]. Similarly, the problem of fundamental limitations in linear and nonlinear consensus networks with stochastic interactions among network components are addressed in [26]–[31]. A small gain theorem for MIMO linear systems with multiplicative noise in mean square sense is given in [32]. In [33], the author considers the discrete-time system with correlated stochastic uncertainties and develops necessary, sufficient conditions for mean square exponential stability expressed in terms of spectral radius of input-output linear matrix operator. However, all the above results are developed for discrete-time network dynamical systems.

The results in this paper are inspired from [19] and can be viewed as a continuous-time counterpart of the discrete-time results developed in [19]. Following [19], we provide a robust control-based framework for the analysis and synthesis of continuous-time linear networked systems with stochastic channel uncertainties. The main contributions of this paper can be stated as follows. We provide a necessary, sufficient condition for mean square exponential stability of continuous-time linear networked system with input and output channel uncertainties. The necessary, sufficient conditions for mean square exponential stability are stated in the form of a spectral radius involving mean square system norm.

We show the mean square system norm introduced in [19] for discrete-time system generalizes to continuous-time setting. LMI-based optimization formulation is proposed for the computation of the mean square system norm. The LMI-based optimization formulation is used for the synthesis of a dynamic controller robust to input and output measurement noise. Furthermore, fundamental limitation result for mean square exponential stabilization expressed in terms of the unstable eigenvalues of the open-loop system is derived. One of the main differences between the discrete-time problem set-up discussed in [19] and continuous-time problem set-up is that, we assume the plant dynamics to be strictly proper. The assumption is necessary to avoid two white noise processes from multiplying each other when the signal traverse in the feedback loop. Furthermore by adopting density-based deterministic approach involving Fokker-Planck equation [34], we avoid the technical challenge associated in dealing with stochastic calculus of stochastic differential equation.

Organization of the paper is as follows. In Section II, we recall some fundamentals of stochastic continuous-time LTI systems with multiplicative noise and provide the derivation for covariance propagation equation. Mean square exponential stability analysis for the continuous-time stochastic systems, based on a mean square system norm, is discussed in Section III-A. Using LMI techniques, controller synthesis for a continuous-time LTI system with stochastic uncertainty is formulated in Section III-B. In Section IV, we discuss the fundamental limitations that arise in the stabilization of continuous-time LTI systems with stochastic uncertainty in feedback loops. Simulation results on a dynamic power network are presented in Section V. Finally, we end the paper with concluding remarks in Section VI.

## II. PRELIMINARIES AND DEFINITIONS

In this section, we provide the preliminaries behind the density-based approach for the analysis of stochastic differential equations (SDE) and stochastic stability definitions used in this manuscript. Consider the following linear stochastic differential equation with stochastic multiplicative uncertainty,

$$dx(t) = Ax(t)dt + \sum_{k=1}^p \sigma_k B_k x(t) d\Delta_k(t), \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $\Delta_1(t), \dots, \Delta_p(t)$  are an independent standard Wiener process, and  $\sigma_k > 0$  are positive constants. The processes,  $\Delta_1(t), \dots, \Delta_p(t)$  are standard independent Wiener processes and for  $i = 1, \dots, p$ , they satisfy

- (i)  $\text{Prob}\{\Delta_i(0) = 0\} = 1$ .
- (ii)  $\{\Delta_i(t)\}$  is a process with independent increments.
- (iii)  $\{\Delta_i(t) - \Delta_i(s)\}$  has a Gaussian distribution with  $E[\Delta_i(t) - \Delta_i(s)] = 0$  and  $E[(\Delta_i(t) - \Delta_i(s))^2] = |t - s|$ .

The following definition for mean square exponential stability can be stated for system given in Eq. (1).

*Definition 1:* [Mean Square Exponentially Stable] System (1) is mean square exponentially stable, if there exist positive constants  $K$  and  $\beta$ , such that

$$E[x(t)^\top x(t)] \leq K \exp^{-\beta t} E[x(0)^\top x(0)] \quad \forall x(0) \in \mathbb{R}^n.$$

We now consider following SDE with both multiplicative and additive stochastic uncertainty,

$$dx(t) = Ax(t)dt + \sum_{k=1}^p \sigma_k B_k x(t) d\Delta_k(t) + Gdw(t), \quad (2)$$

where  $w(t) \in \mathbb{R}^n$  is a standard Wiener process and is assumed uncorrelated with  $\Delta_1(t), \dots, \Delta_p(t)$ . We now define the following notion of bounded moment stability for system (2).

*Definition 2 (Second Moment Bounded):* System (2) is said to be second moment bounded if there exists a positive constant  $\bar{K}$ , such that  $\lim_{t \rightarrow \infty} E[x(t)^\top x(t)] \leq \bar{K}$  for all  $x(0) \in \mathbb{R}^n$ .

Let  $x(t)$  be the solution of system (2). The density function  $u(x, t)$  for the process  $x(t)$  is uniquely defined [34] and satisfies  $\text{Prob}\{x(t) \in B\} = \int_B u(z, t) dz$  for any set  $B \subset \mathbb{R}^n$ . The density function  $u(x, t)$  is obtained as a solution to the Fokker-Planck (FP) equation, also called the Kolmogorov forward equation [34, Thm. 11.6.1]. The FP equation is defined as follows

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \left( \sum_{k=1}^p \sigma_k^2 (b_k^i x)(b_k^j x) + G_i G_j \right) u - \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i x) u, \quad t > 0, x \in \mathbb{R}^n, \quad (3)$$

where  $a_i, b_k^i$  are the  $i^{th}$  rows of  $A, B_k$  respectively. Further, it is known for the case of a linear system driven by the additive white noise process, if the initial density function,  $u(x, 0)$ , is Gaussian, then  $u(x, t)$  remains Gaussian for all future time  $t$ . Hence, for linear systems with additive white noise forcing, the infinite dimensional FP equation can be replaced with the

finite dimensional equation for the evolution of the mean and covariance. In the following lemma, we show the covariance evolution for system (2) is closed and does not depend upon higher order moments.

*Lemma 3:* Let the covariance matrix  $\bar{Q}(t) = E[x(t)x(t)^\top | u] := \int_{\mathbb{R}^n} xx^\top u(x, t) dx$ , then  $\bar{Q}(t)$  satisfies the following matrix differential equation (MDE) for system (2),

$$\dot{\bar{Q}} = \bar{Q}A^\top + A\bar{Q} + \sum_{k=1}^p \sigma_k^2 B_k \bar{Q} B_k^\top + GG^\top \quad (4)$$

and the covariance equation for system (1) satisfies

$$\dot{Q} = QA^\top + AQ + \sum_{k=1}^p \sigma_k^2 B_k Q B_k^\top. \quad (5)$$

*Proof:* Consider,  $V(x) > 0$  and  $V(0) = 0$  only for  $x = 0$ . Then,  $E[V|u] = \int_{\mathbb{R}^n} V(x(t))u(x, t) dx$ . Taking the derivative on both sides, we obtain [34, Theorem 11.9.1]

$$\begin{aligned} \frac{dE[V|u]}{dt} &= \int_{\mathbb{R}^n} \left\{ \frac{1}{2} \sum_{i,j=1}^n \left[ \sum_{k=1}^p \sigma_k^2 (b_k^i x)(b_k^j x) + G_i G_j \right] \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^n (a_i x) \frac{\partial V}{\partial x_i} \right\} u(x, t) dx \\ &:= \int_{\mathbb{R}^n} \mathcal{L}V u(x, t) dx := E[\mathcal{L}V|u]. \end{aligned} \quad (6)$$

Now, consider,  $V(x(t)) = x(t)^\top P x(t)$ , where  $P = P^\top > 0$ . Then,

$$\mathcal{L}V(x(t)) = x(t)^\top \left( A^\top P + PA + \sum_{k=1}^p \sigma_k^2 B_k^\top P B_k \right) x(t) + G^\top P G. \quad (7)$$

Substituting Eq. (7) in Eq. (6), using the linearity of trace, expectation and commutativity inside trace, we obtain,

$$\frac{d(\text{tr}(E[xx^\top | u]P))}{dt} = \text{tr} \left( (A^\top P + PA + \sum_{k=1}^p \sigma_k^2 B_k^\top P B_k) E[xx^\top | u] + GG^\top P \right).$$

By definition of expectation,  $E[xx^\top | u] = \bar{Q}$  and we have,

$$\text{tr}(\dot{\bar{Q}}P) = \text{tr} \left( (\bar{Q}A^\top + A\bar{Q} + \sum_{k=1}^p \sigma_k^2 B_k \bar{Q} B_k^\top + GG^\top) P \right).$$

This can be rewritten in terms of an inner product as

$$\langle \dot{\bar{Q}} - (\bar{Q}A^\top + A\bar{Q} + \sum_{k=1}^p \sigma_k^2 B_k \bar{Q} B_k^\top + GG^\top), P \rangle = 0.$$

Since  $P > 0$ ,

$$\dot{\bar{Q}} = \bar{Q}A^\top + A\bar{Q} + \sum_{k=1}^p \sigma_k^2 B_k \bar{Q} B_k^\top + GG^\top.$$

By substituting  $G = 0$ , we obtain the covariance propagation equation for the system without additive noise as shown in Eq. (1). ■

*Lemma 4:* The system (1) is mean square exponentially stable if and only if system (2) is second moment bounded.

*Proof:* Using the operator  $\phi$ , that transforms a matrix into a vector as defined in [35][Chapter 2], the MDE's given in Eq. (5) and Eq. (4) are written as linear differential equations as given below.

$$\dot{q} = \mathcal{A}q, \quad (8)$$

$$\dot{\bar{q}} = \mathcal{A}\bar{q} + \mathcal{B}, \quad (9)$$

where  $q = \phi(Q)$ ,  $\bar{q} = \phi(\bar{Q})$ ,  $\mathcal{B} = (G \otimes G)\phi(I) \in \mathbb{R}^{n^2}$  and  $\mathcal{A} = A \oplus A + \sum_{k=1}^p \sigma_k^2(B_k \otimes B_k) \in \mathbb{R}^{n^2 \times n^2}$ , where  $I$  is the identity matrix of size  $n \times n$  and  $\otimes$  denotes the Kronecker product,  $\oplus$  is the Kronecker sum.

*Necessity:* The mean square exponential stability of system (1) yields stability of system (8), that is,  $\mathcal{A}$  is Hurwitz. Since  $\mathcal{A}$  is Hurwitz, the steady state value of  $\bar{q}$  is given by  $\lim_{t \rightarrow \infty} \bar{q}(t) = \lim_{t \rightarrow \infty} \phi(\bar{Q}(t)) = -\mathcal{A}^{-1}\mathcal{B}$ . Now, taking the inverse  $\phi$  operator, we obtain,  $\lim_{t \rightarrow \infty} E[x(t)x(t)^\top] = -\phi^{-1}(\mathcal{A}^{-1}\mathcal{B})$ , where  $\phi^{-1}(\mathcal{A}^{-1}\mathcal{B})$  is finite. Therefore, system (2) is second moment bounded.

*Sufficiency:* If system (2) is second moment stable, then  $\lim_{t \rightarrow \infty} \bar{Q}(t)$  is a finite value. Taking the operator, it can be alternately written as,  $\lim_{t \rightarrow \infty} \phi(\bar{Q}(t)) = \lim_{t \rightarrow \infty} e^{\mathcal{A}t}\phi(\bar{Q}(0)) - \phi^{-1}(\mathcal{A}^{-1}\mathcal{B})$ . The limit on the right-hand side is finite, if and only if  $\mathcal{A}$  is Hurwitz, which implies system (8) is stable and hence system (1) is mean square exponentially stable. ■

### III. MAIN RESULTS

The main results of this paper can be explained in the context of the block diagram as shown in Fig. 1. The set-up follows closely the one used in [19] for mean square exponential stability analysis of a discrete-time network. In Fig. 1, we show the feedback interconnection of plant and controller with stochastic uncertainty in the input and output loops.

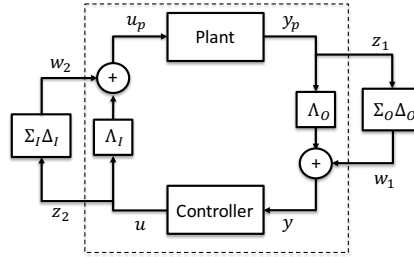


Fig. 1: Mean and uncertain part of the MIMO system

The plant dynamics are described as

$$\mathbb{P} : \begin{cases} \dot{x}_p &= A_p x_p + B_p u_p \\ y_p &= C_p x_p \end{cases}, \quad (10)$$

where  $x_p \in \mathbb{R}^n$ ,  $u_p \in \mathbb{R}^d$ , and  $y_p \in \mathbb{R}^q$  are the plant state, input, and output, respectively. It is assumed, the state space model for the plant is stabilizable, detectable, and strictly proper. Similarly, the controller dynamics are assumed to be strictly proper with the following state space model,

$$\mathbb{K} : \begin{cases} \dot{x}_k &= A_k x_k + B_k y_k \\ u_k &= C_k x_k \end{cases}, \quad (11)$$

where  $x_k \in \mathbb{R}^n$ ,  $y_k \in \mathbb{R}^q$ ,  $u_k \in \mathbb{R}^d$ . The assumption on the controller dynamics being strictly proper is essential, since it allows us to study the case where the uncertainties enter both at the input and output channels. If the uncertainty enters only at the input or the output channel (refer to Fig. 1), then one can consider the controller dynamics which is not strictly proper [36]. The plant output is fed to the controller through the stochastic uncertainty block, modeled as follows,

$$y_k = \left( \Lambda_O + \Sigma_O \frac{d\Delta_O}{dt} \right) y_p, \quad (12)$$

where  $\Lambda_O := \text{diag}\{\lambda_O^1, \dots, \lambda_O^q\} \in \mathbb{R}^q$  is the mean of the uncertainty,  $\Sigma_O := \text{diag}\{\sigma_O^1, \dots, \sigma_O^q\} \in \mathbb{R}^q$  is the standard deviation of the uncertainty, and  $\frac{d\Delta_O}{dt} := \text{diag}\{\frac{d\Delta_O^1}{dt}, \dots, \frac{d\Delta_O^q}{dt}\}$  with  $\Delta_O^1(t), \dots, \Delta_O^q(t)$  being the independent standard Wiener processes.

Similarly, the input channel uncertainty model is,

$$u_p = \left( \Lambda_I + \Sigma_I \frac{d\Delta_I}{dt} \right) u_k, \quad (13)$$

where  $\Lambda_I = \text{diag}\{\lambda_I^1, \dots, \lambda_I^d\} \in \mathbb{R}^d$  is the mean value of the input uncertainty,  $\Sigma_I = \text{diag}\{\sigma_I^1, \dots, \sigma_I^d\} \in \mathbb{R}^d$  is the standard deviation of the uncertainty, and  $\frac{d\Delta_I}{dt} = \text{diag}\{\frac{d\Delta_I^1}{dt}, \dots, \frac{d\Delta_I^d}{dt}\}$  with  $\Delta_I^1(t), \dots, \Delta_I^d(t)$  being the independent standard Wiener processes. Both the input and output channel uncertainties are assumed to be uncorrelated.

The mean part of the input and output uncertainties can be separated from the zero mean part to define a nominal or mean network as shown within the dotted box in Fig. 1. The zero mean part of the stochastic uncertainty now interacts with the nominal or mean network dynamics through disturbance signals,  $w_1 \in \mathbb{R}^q$  and  $w_2 \in \mathbb{R}^d$ . The network system can be written in a standard robust control form by defining the control variable signals,  $z_1 \in \mathbb{R}^q$  and  $z_2 \in \mathbb{R}^d$ . With reference to the block diagram in Fig. 1, the control variable signals are given by,  $z_1 = y_p = C_p x_p$ ,  $z_2 = u_k = C_k x_k$ . Similarly, the disturbance signals are given by  $w_1 = \Sigma_O \frac{d\Delta_O}{dt} z_1$ ,  $w_2 = \Sigma_I \frac{d\Delta_I}{dt} z_2$ .

We re-enumerate the input and output uncertainties,  $\Delta_I^1(t), \dots, \Delta_I^d(t)$ ,  $\Delta_O^1(t), \dots, \Delta_O^q(t)$  as  $\Delta_1, \Delta_2, \dots, \Delta_m$ , where  $m = d + q$ . With the definition of new control and disturbance variables, the feedback interconnection of the nominal plant and the controller is denoted by  $G = \mathcal{F}(\mathbb{P}, \mathbb{K})$ . This feedback interconnection has the following state space form.

$$G : \begin{cases} \dot{x} &= Ax + Bw \\ z &= Cx \end{cases}, \quad (14)$$

where  $x = (x_p^\top, x_k^\top)^\top \in \mathbb{R}^{2n}$ ,  $z = (z_1^\top, z_2^\top)^\top \in \mathbb{R}^m$ ,  $w = (w_1^\top, w_2^\top)^\top \in \mathbb{R}^m$ ,  $C = \text{diag}(C_p, C_k)$ ,

$$A = \begin{pmatrix} A_p & B_p \Lambda_I C_k \\ B_k \Lambda_O C_p & A_k \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_p \\ B_k & 0 \end{pmatrix}.$$

We now introduce the following definition of mean square norm for system  $G$  [19].

*Definition 5 (Mean Square Norm):* The mean square norm for system  $G$  is defined as follows,

$$\|G\|_{MS} = \max_{k=1, \dots, m} \sqrt{\sum_{\ell=1}^m \|G_{k\ell}\|_2^2},$$

where  $\|G_{kl}\|_2$  is the standard  $\mathcal{H}_2$  norm of the system with input  $l$  and output  $k$ .

The feedback interconnection of the nominal system  $G$  with the uncertainty  $\Delta$  (with some abuse of notation) is denoted by  $\mathcal{F}(G, \Delta)$  and has the following state space representation.

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \\ w &= \Delta z := \begin{pmatrix} \Sigma_O \frac{d\Delta_O}{dt} & 0 \\ 0 & \Sigma_I \frac{d\Delta_I}{dt} \end{pmatrix} z \end{aligned} \quad (15)$$

*Remark 6:* Note that, although we arrive at system (15) given in standard robust control form with input and output channel uncertainties, the framework is general enough to model stochastic parametric uncertainty in system plant,  $A_p$ , matrix.

We make the following assumptions on this feedback interconnected system.

*Assumption 7:* (a) The deterministic system (14) denoted by  $G$  is internally stable, that is,  $A$  is Hurwitz and moreover,  $G$  is considered to be strictly proper.

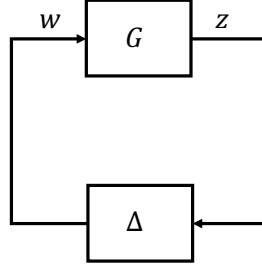


Fig. 2: Feedback interconnection of  $G$  &  $\Delta$ ,  $\mathcal{F}(G, \Delta)$ .

- (b) The initial state of the system  $G$ , denoted by  $x(0)$  has bounded variance and is independent from  $\Delta_i(t)$  for each  $i \in \{1, \dots, m\}$ .

In the next section, we discover the analysis results providing the necessary, sufficient conditions for mean square exponential stability of nominal system  $G$  in feedback interconnection with uncertainty  $\Delta$ . The stochastic interconnected system (15) can be written as system (1) for which the mean square exponential stability applies and is given in Definition 1.

#### A. Analysis

The following theorem provides necessary and sufficient conditions for the mean square exponential stability of the interconnected system,  $G$  and  $\Delta$  i.e.,  $\mathcal{F}(G, \Delta)$ .

*Theorem 8:* Under Assumption 7, the feedback interconnected system (15) shown in Fig. 2 is mean square exponentially stable, if and only if, there exists a  $P > 0$ , such that, it satisfies

$$A^\top P + PA + \sum_{i=1}^m \sigma_i^2 C_i^\top B_i^\top P B_i C_i < 0. \quad (16)$$

*Proof: Sufficiency:* The covariance propagation equation for the feedback interconnected system with uncertainty is

$$\dot{Q}(t) = Q(t)A^\top + AQ(t) + \sum_{i=1}^m \sigma_i^2 B_i C_i Q(t) C_i^\top B_i^\top. \quad (17)$$

This covariance propagation equation is a matrix differential equation and follows from Lemma 3. To achieve mean square exponential stability,  $Q(t)$  should converge to zero exponentially. To show this, we construct the Lyapunov function  $V(Q(t)) = \text{tr}(Q(t)P)$ , where  $P > 0$ . Then,

$$\dot{V}(Q(t)) = \text{tr}\left((Q(t)A^\top + AQ(t) + \sum_{i=1}^m \sigma_i^2 B_i C_i Q(t) C_i^\top B_i^\top)P\right).$$

Further,  $\dot{V}(Q(t)) = \text{tr}(-Q(t)M)$  for some positive matrix  $M > 0$ . Since  $M > 0$ , there exists an  $\alpha = \frac{\lambda_{\min}(M)}{\lambda_{\max}(P)} > 0$ , such that  $\alpha P \leq M$ . Therefore,  $\dot{V}(Q(t)) \leq -\alpha V(Q(t))$  and system (15) is mean square exponentially stable.

*Necessity:* Let  $\phi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$  be a bijective operator [35, Chapter 2] which converts a matrix into a column vector. Assume system (15) to be mean square exponentially stable. Then, we know the covariance matrix,  $Q(t)$ , converges exponentially to zero. This implies, we have a stable evolution for  $q(t) = \phi(Q(t)) \in \mathbb{R}^{n^2}$  with the following dynamics,

$$\dot{q}(t) = \mathcal{A}q(t), \quad (18)$$

where  $\mathcal{A} = A \oplus A + \sum_{i=1}^m \sigma_i^2 (B_i C_i \otimes B_i C_i)$ . Stability of system (18) implies  $\mathcal{A}$  is Hurwitz and hence  $\mathcal{A}^\top$  is also Hurwitz. Therefore, the evolution,  $\dot{r}(t) = \mathcal{A}^\top r(t)$  is stable and satisfies the following matrix differential equation,

$$\dot{R}(t) = A^\top R(t) + R(t)A + \sum_{i=1}^m \sigma_i^2 C_i^\top B_i^\top R(t) B_i C_i \quad (19)$$

which is also stable. Let  $R(0) > 0$ , and  $R(t)$  denote the solution for Eq. (19). Since  $R(t)$  satisfies the stable first order linear differential equation, the function  $P(t) = \int_0^t R(\tau) d\tau$  has a finite value. Integrating on both sides of Eq. (19) and simplifying, we obtain

$$\dot{P}(t) - R(0) = A^\top P(t) + P(t)A + \sum_{i=1}^m \sigma_i^2 C_i^\top B_i^\top P(t) B_i C_i.$$

Observing that Eq. (19) is stable, as  $t \rightarrow \infty$ , we obtain,

$$A^\top P + PA + \sum_{i=1}^m \sigma_i^2 C_i^\top B_i^\top P B_i C_i = -R(0),$$

where  $P := \lim_{t \rightarrow \infty} P(t)$ . The result now follows by noticing that  $R(0) > 0$ . ■

The ensuing result gives an alternative representation of the inequality given in Eq. (16), which is helpful in writing the LMI-based optimization for computing the mean square norm of system  $G$ . The following lemmas, theorems, and their proofs can be viewed as continuous-time counterpart of the discrete-time results from [19].

*Lemma 9:* The inequality Eq. (16) holds, if and only if, there exists a  $\mathcal{Q} > 0$ , and  $\alpha_i > 0$  for every  $i = 1, 2, \dots, m$ , such that

$$\begin{aligned} A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m B_i \alpha_i B_i^\top &< 0, \\ \alpha_i &> \sigma_i^2 C_i \mathcal{Q} C_i^\top, \quad i = 1, 2, \dots, m. \end{aligned} \quad (20)$$

*Proof:* The dual inequality equivalent to Eq. (16) is

$$A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m \sigma_i^2 B_i C_i \mathcal{Q} C_i^\top B_i^\top < 0, \quad (21)$$

where  $\mathcal{Q} > 0$ . Observe the straight forward substitution leads to a sufficiency condition. In showing the necessary part, for some matrix  $M > 0$ , the inequality (21) can be rewritten as

$$A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m \sigma_i^2 B_i C_i \mathcal{Q} C_i^\top B_i^\top + M = 0. \quad (22)$$

Since we have,  $\sum_{i=1}^m B_i B_i^\top \geq 0$ , choose  $\epsilon > 0$ , such that

$$0 \geq \epsilon \sum_{i=1}^m B_i B_i^\top < M.$$

Using this in Eq. (22), we get

$$\begin{aligned} A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m \sigma_i^2 B_i C_i \mathcal{Q} C_i^\top B_i^\top + \epsilon \sum_{i=1}^m B_i B_i^\top &< 0 \\ \Rightarrow A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m (\epsilon + \sigma_i^2 C_i \mathcal{Q} C_i^\top) B_i B_i^\top &< 0. \end{aligned}$$

Define  $\alpha_i = \epsilon + \sigma_i^2 C_i \mathcal{Q} C_i^\top$  and we obtain,

$$\begin{aligned} A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m B_i \alpha_i B_i^\top &< 0, \\ \alpha_i &> \sigma_i^2 C_i \mathcal{Q} C_i^\top \quad \text{for } i = 1, 2, \dots, m. \end{aligned}$$



■

In the ensuing result, a LMI-based optimization formulation is provided for the computation of mean square system norm.

*Lemma 10:* Suppose  $A$  is Hurwitz and let  $\theta > 0$  be a diagonal matrix. Then, we obtain,

$$\begin{aligned} \|\theta^{-1}G\theta\|_{MS}^2 &= \inf_{\mathcal{P}>0, \mathcal{S}>0, \gamma} \gamma \\ \text{subject to} \\ &\begin{bmatrix} A^\top \mathcal{P} + \mathcal{P}A & \mathcal{P}B\theta \\ \theta B^\top \mathcal{P} & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} \theta \mathcal{S}\theta & C \\ C^\top & \mathcal{P} \end{bmatrix} > 0, \\ &\mathcal{S}_{ii} < \gamma, \quad i = 1, 2, \dots, m. \end{aligned}$$

*Proof:* The idea of the proof is as follows. First, we formulate the computation of optimal  $\mathcal{H}_2$  norm as an optimization problem. By noticing the differences between the  $\mathcal{H}_2$  norm and the mean square norm, the result follows. For the system,  $\theta^{-1}G\theta$ , the mean square exponential stability conditions can be equivalently written as

$$A\mathcal{Q} + \mathcal{Q}A^\top + \sum_{i=1}^m B_i \theta_i^2 B_i^\top < 0, \quad (23)$$

$$\gamma_i \theta_{ii}^2 > C_i \mathcal{Q} C_i^\top, \quad i = 1, 2, \dots, m, \quad (24)$$

where  $\theta_{ii}$ 's are the diagonal elements of  $\theta$ . By noticing the  $B_i$ 's are the columns of  $B$  matrix and multiplying with  $\mathcal{P} = \mathcal{Q}^{-1}$  on both sides of inequality (23), we obtain

$$A^\top \mathcal{P} + \mathcal{P}A + \mathcal{P}B\theta\theta^\top B^\top \mathcal{P}^\top < 0. \quad (25)$$

Further, the elementwise inequalities Eq. (24) can be written as a linear matrix inequality as given below.

$$\begin{aligned} \theta \mathcal{S} \theta &> C \mathcal{P}^{-1} C^\top \\ \mathcal{S}_{ii} &< \gamma_i, \quad i = 1, 2, \dots, m. \end{aligned} \quad (26)$$

Now rewriting Eqs. (25), (26) using Schur compliments, the computation for  $\mathcal{H}_2$  norm problem can be written as an LMI optimization problem as shown below.

$$\begin{aligned} \|\theta^{-1}G\theta\|_2^2 &= \inf_{\gamma_i, \mathcal{S}>0, \mathcal{P}>0} \sum_i \gamma_i \\ \text{subject to} \\ &\begin{bmatrix} A^\top \mathcal{P} + \mathcal{P}A & \mathcal{P}B\theta \\ \theta B^\top \mathcal{P} & -I \end{bmatrix} < 0 \\ &\begin{bmatrix} \theta \mathcal{S} \theta & C \\ C^\top & \mathcal{P} \end{bmatrix} > 0 \\ &\mathcal{S}_{ii} < \gamma_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

This computation of  $\mathcal{H}_2$  norm problem as an LMI optimization problem is not new and has been discussed in [1], [37]. A similar result for the discrete-time case has been discussed in [19]. Now, the cost is modified to obtain the result by observing the difference between  $\|G\|_{MS}^2 = \max_{i=1:m} \mathcal{S}_{ii}$  and  $\|G\|_2^2 = \sum_{i=1}^m \mathcal{S}_{ii}$ . ■

The following theorem is the main result of this section and provides equivalent necessary, sufficient conditions for mean square exponential stability of feedback interconnected system,  $\mathcal{F}(G, \Delta)$ . In fact, the results of the following theorem can be viewed as a stochastic counterpart of the small gain theorem for the continuous-time system.

*Theorem 11:* Consider the feedback interconnection  $\mathcal{F}(G, \Delta)$  as shown in Fig. 2. Let system  $G$  and uncertainty  $\Delta$  satisfy Assumption 7. Then, the following stability conditions for mean square exponential stability are equivalent.

- (a) The feedback interconnection of the system  $G$  with  $\Delta$  is mean square exponentially stable.
- (b) There exists a  $\mathcal{Q} > 0$  and  $\alpha_i > 0$ , for every  $i = 1, \dots, m$  satisfying the LMI given in Eq. (20).
- (c)  $\rho(\tilde{G}\tilde{\Sigma}) < 1$ , where  $\rho$  stands for the spectral radius of a matrix and  $\tilde{\Sigma} = \text{diag}(\sigma_1^2, \dots, \sigma_m^2)$ ,

$$\tilde{G} = \begin{pmatrix} \|G_{11}\|_2^2 & \dots & \|G_{1m}\|_2^2 \\ \vdots & \ddots & \vdots \\ \|G_{m1}\|_2^2 & \dots & \|G_{mm}\|_2^2 \end{pmatrix}.$$

The notation,  $\|G_{ij}\|_2$  is the  $\mathcal{H}_2$  norm of the system from disturbance input,  $j$  and controlled output,  $i$ .

Further, for  $\sigma_1^2 = \dots = \sigma_m^2 = \sigma^2$ , the feedback interconnection is mean square exponentially stable if and only if

$$\sigma^2 \inf_{\theta > 0, \theta - \text{diag}} \|\theta^{-1}G\theta\|_{MS}^2 < 1.$$

*Proof:* (a)  $\Leftrightarrow$  (b) This follows by combining the results from Theorem 8 and Lemma 9.

(b)  $\Leftrightarrow$  (c) This result follows by distributing the system to single input single output systems and using the spectral radius definition of nonnegative matrices discussed in [38].

In the special case of all variances to be the same, the result follows from Lemma 10 and by choosing

$$\theta = \text{diag}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots, \sqrt{\alpha_p}).$$

■

Here, the mean square norm is computed for the transformed system  $\theta^{-1}G\theta$ . The scaling factor,  $\theta$ , ensures the mean square norm with respect to all inputs and outputs is same. Hence, for a SISO system, the scaling factor,  $\theta$ , does not come into play. Moreover, in the case for a SISO system, the mean square norm is equal to the standard  $\mathcal{H}_2$  norm.

*Remark 12:* The equivalent condition (c) from Theorem 11 can be used to determine the maximum tolerable variance of uncertainty  $\sigma^*$  above, where the feedback interconnection will be mean square exponentially unstable. In particular, the critical  $\sigma^*$  is given by  $\sigma^* = \frac{1}{\sqrt{\inf_{\theta > 0, \theta - \text{diag}} \|\theta^{-1}G\theta\|_{MS}^2}}$ .

In the next section, we discuss the controller synthesis formulation when all the uncertainties in the input and output channels are considered to be same, i.e.,  $\sigma_1^2 = \dots = \sigma_m^2 = \sigma^2$ .

## B. Controller synthesis

In this section, we tackle the controller synthesis problem for the closed-loop system,  $G = \mathcal{F}(\mathbb{P}, \mathbb{K})$  with stochastic uncertainty,  $\Delta$  in the feedback. The controller is designed such that the closed-loop system can tolerate maximum uncertainty. Using part (c) of Theorem 11 and Lemma 10, we pose the controller synthesis problem as an LMI-based optimization problem,  $\inf_{\mathbb{K} - \text{stab}, LTI} \inf_{\theta > 0, \text{diag}} \|\theta^{-1}\mathcal{F}(\mathbb{P}, \mathbb{K})\theta\|_{MS}^2$ .

This optimization provides a robust optimal controller by searching in the space of linear time invariant stabilizing controllers that minimizes the mean square norm. However, searching for a robust optimal controller is a nonconvex problem. This problem

can be made convex by following the approach given in [37] along with fixing the variable  $\theta$ . Later, in simulations, we solve the optimization problem for the controller by keeping the variable  $\theta$  constant which results in just a robust controller. The resultant controller formulation is given in the ensuing theorem.

*Theorem 13:* Given a plant  $\mathbb{P}$  and for any  $\theta > 0$ , the optimization problem:  $\inf_{\mathbb{K}-stab, LTI} \|\theta^{-1}G\theta\|_{MS}^2$  is equivalent to the following LMI optimization:

$$\begin{aligned} & \inf_{\mathbf{X}, \mathbf{Y}, \mathbf{S}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \gamma} \gamma \\ & \text{subject to} \\ & \begin{bmatrix} A_p \mathbf{X} + \mathbf{X} A_p^\top + B_p \Lambda_I \hat{\mathbf{C}} + (B_p \Lambda_I \hat{\mathbf{C}})^\top & \hat{\mathbf{A}}^\top + A_p & 0 & B_p \\ & \hat{\mathbf{A}} + A_p^\top & A_p^\top \mathbf{Y} + \mathbf{Y} A_p + \hat{\mathbf{B}} \Lambda_O C_p + (\hat{\mathbf{B}} \Lambda_O C_p)^\top & \hat{\mathbf{B}} & \mathbf{Y} B_p \\ & 0 & \hat{\mathbf{B}}^\top & & \\ & B_p^\top & (\mathbf{Y} B_p)^\top & & -\theta^{-2} \end{bmatrix} < 0, \\ & \begin{bmatrix} \theta \mathbf{S} \theta & C_p \mathbf{X} & C_p \\ & \hat{\mathbf{C}} & 0 \\ (C_p \mathbf{X})^\top & \hat{\mathbf{C}}^\top & \mathbf{X} & I \\ C_p^\top & 0 & I & \mathbf{Y} \end{bmatrix} > 0, \\ & S_{ii} < \gamma, \quad i = 1, 2, \dots, m, \end{aligned}$$

where  $\mathbf{X}, \mathbf{Y}, \mathbf{S}$  are positive definite symmetric matrices of size  $n \times n, n \times n, m \times m$ , and  $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$  are matrices of sizes  $n \times n, n \times q, d \times n$  correspondingly. A feasible solution to the above optimization is a controller of the order of the plant,  $\mathbb{P}$ . Then, the system matrices of the controller can be uniquely obtained as follows:

$$\begin{aligned} C_K &= \hat{\mathbf{C}}(M^\top)^{-1}, \\ B_K &= N^{-1} \hat{\mathbf{B}}, \\ A_K &= N^{-1}(\hat{\mathbf{A}} - \mathbf{Y} A_p \mathbf{X} - N B_K \Lambda_O C_p \mathbf{X} - \mathbf{Y} B_p \Lambda_I C_K M^\top)(M^\top)^{-1}, \end{aligned}$$

where  $M, N$  are invertible matrices satisfying  $N M^\top = I - \mathbf{Y} \mathbf{X}$ . One possible choice for  $N$  is  $N N^\top = \mathbf{Y} - \mathbf{X}^{-1}$  and  $M$ , such that

$$\begin{pmatrix} \mathbf{Y} & N \\ N^\top & I \end{pmatrix} \begin{pmatrix} \mathbf{X} & M \\ M^\top & * \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

*Proof:* The result follows from Lemma 10 and applying congruence transformation as shown in [37]. ■

A similar result on controller synthesis in the case of a discrete-time system with uncertainty in feedback communication channels is given in [19]. Furthermore, in [19], the author briefly mentions different ways to approach this type of nonconvex problem. One of the ways to solve the controller synthesis problem is by applying sub-optimal methods, such as the D-K iteration [1]. In this approach, first  $\theta$  is fixed to solve for the controller matrices and then  $\theta$  is updated by keeping the controller matrices constant. This process is continued until the update equation for  $\theta$  converges. In general, this approach does not guarantee a global optimal controller, but can always provide a local optimal controller.

#### IV. FUNDAMENTAL LIMITATIONS

The results derived in the previous section provide a framework for determining the largest variance of channel uncertainty. However, the variance value itself must be computed numerically. We do not have the analytical expression for the largest variance value expressed in terms of characteristics of the open-loop system dynamics. Later, in the simulation section, we show the variance value is a function of both the open-loop unstable poles and zeros. In this section, we will consider a special case of single input and full state feedback. The channel uncertainty is assumed at the input side only. With single uncertainty in the feedback loop, the mean square system norm is reduced to standard  $\mathcal{H}_2$  norm. Furthermore, using the standard results from robust control theory [39], we know using the full state feedback measurements, the optimal  $\mathcal{H}_2$  performance obtained from static and dynamic controllers are the same. Hence, to find the controller giving optimal  $\mathcal{H}_2$  norm, it is enough to restrict the search to the class of static controllers. With some abuse of notation, we write the single-input LTI system with input channel uncertainty as follows.

$$\begin{aligned}\dot{x} &= A_o x + B u, \quad v = K x \\ u &= (\mu + \sigma d\delta) v,\end{aligned}\tag{27}$$

where  $u \in \mathbb{R}$  and  $\mu + \sigma d\delta$  models the channel uncertainty with mean value,  $\mu \neq 0$ , variance,  $\sigma^2$ , and  $d\delta$  is the white noise process. System matrix,  $A_o$  correspond to the open-loop system. We now make the following assumption.

*Assumption 14:* Assume all the eigenvalues of  $A_o$  are in the right-half plane, i.e.,  $-A_o$  is Hurwitz and the pair  $(A_o, B)$  is stabilizable.

Since,  $A_o$  has all eigenvalues on the right hand side, the stabilizability of pair  $(A_o, B)$  is equivalent to controllability of pair  $(A_o, B)$ . Moreover, this implies pair  $(A_o, \mu B)$  is also controllable and there exists a stabilizing controller  $K$  such that  $A := A_o + \mu B K$  is Hurwitz. The objective here is to design a state feedback controller, so the closed-loop system is mean square exponentially stable with maximum tolerable variance,  $\sigma_*^2$ . The closed-loop system is written as follows.

$$dx = A x dt + \sigma B K x d\delta.\tag{28}$$

where the closed-loop system matrix,  $A := A_o + \mu B K$ .

*Theorem 15:* Consider the stabilization problem for single-input full state feedback LTI system with channel uncertainty at the input side shown in Eq. (27). Under Assumption 14, system (27) is mean square exponentially stable, if and only if,  $2 \frac{\sigma^2}{\mu^2} \sum_i \lambda_i(A_o) < 1$ .

*Proof:* The closed-loop system (28) can be written as a SISO system,  $G$ , whose system matrices are  $G = \begin{pmatrix} A_o + \mu B K & B \\ K & 0 \end{pmatrix}$ .

This follows from noticing that the input and output are related by  $u = \sigma d\delta y$  and  $y = K x$ . We know that the mean square norm for a SISO system is equivalent to  $\mathcal{H}_2$  norm of the system.

*Necessity:* From Theorem 11, we know that the necessary condition for mean square exponential stability is,  $\sigma^2 \|G\|_2^2 < 1$ . The  $\mathcal{H}_2$  norm of  $G$ , i.e.,  $\|G\|_2$ , is given by  $B^\top P B$  where  $P > 0$ , and is obtained from

$$(A_o + \mu B K)^\top P + P(A_o + \mu B K) + K^\top K = 0.\tag{29}$$

Now, the optimal  $K$  satisfying Eq. (29) is obtained by minimizing the left hand side (lhs) of Eq. (29), i.e., by taking the derivative of lhs of Eq. (29) w.r.t  $K$  and equating it to 0. This yields,  $K = -\mu B^\top P$  and we have  $A_o^\top P + P A_o - \mu^2 P B B^\top P = 0$ . Further, rewrite this equation by multiplying and dividing the last term with  $\sigma^2 B^\top P B$ , we obtain

$$A_o^\top P + P A_o - \frac{\mu^2 P B B^\top P}{\sigma^2 B^\top P B} \sigma^2 B^\top P B = 0.\tag{30}$$

Now, using the relation,  $\sigma^2 B^\top P B < 1$  from mean square exponential stability, we can rewrite Eq. (30) as

$$A_o^\top P + P A_o - \frac{\mu^2 P B B^\top P}{\sigma^2 B^\top P B} < 0. \quad (31)$$

Since,  $P > 0$ , pre and post multiplying Eq. (31) by  $P^{-\frac{1}{2}}$  on both sides and taking trace, we obtain,  $2\text{tr}(A_o) < \frac{\mu^2}{\sigma^2}$ .

*Sufficiency:* It is enough to show,  $\sigma^2 B^\top P B < 1$ , where  $P > 0$  satisfies Eq. (29). Choose  $K = -\mu B^\top P$ , and rewrite Eq. (29) as

$$A_o^\top P + P A_o - \mu^2 \frac{P B B^\top P}{B^\top P B} B^\top P B = 0. \quad (32)$$

Now, pre and post multiply Eq. (32) by  $P^{-\frac{1}{2}}$  on both sides and take trace. Then, using the fact that  $2\frac{\sigma^2}{\mu^2}\text{tr}(A_o) < 1$ , we obtain,  $\mu^2 B^\top P B < \frac{\mu^2}{\sigma^2}$  and hence the result follows. ■

## V. SIMULATION

In this section, the WSCC 9 bus system is considered to demonstrate the results in this paper. The WSCC 9 bus system with nine buses and three generators (Fig. 3) is an approximation to the Western System Coordinating Council (WSCC). Using the proposed approach, we can determine the critical noise variance that can be tolerated in the communication channels while maintaining the mean square exponential stability. Further, we design a robust wide area controller that can tolerate maximum possible channel uncertainty.

Consider the structure preserving power network model consisting of a linearized swing equation. The resultant dynamic model of the power network is given by

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}\mathcal{L} & -M^{-1}D \end{bmatrix}, \quad (33)$$

where  $\delta = (\delta_1 \ \delta_2 \ \delta_3)^\top$  and  $\omega = (\omega_1 \ \omega_2 \ \omega_3)^\top$  are the generator rotor angles and frequencies respectively. Further, the matrices,  $M$  and  $D$  are diagonal and they represent inertia and damping values of generators. The Laplacian matrix,  $L$  and the Kron-reduced Laplacian matrix,  $\mathcal{L}$  are given by

$$L = \begin{bmatrix} L_{gg} & L_{gl} \\ L_{lg} & L_{ll} \end{bmatrix}; \quad \mathcal{L} = L_{gg} - L_{gl}L_{ll}^{-1}L_{lg}.$$

The elements of the Laplacian matrix corresponding to the power network gives the admittance-weighted interconnections between the generators and load buses. The various parameter values for the 9 bus system are obtained from [40]. The inertia matrix,  $M = \text{diag}\{22.64, 6.47, 5.047\}$ , damping matrix,  $D = \text{diag}\{10, 10, 10\}$  and the Kron-reduced Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} -4.5375 & 2.4111 & 2.4006 \\ 2.4111 & -4.8367 & 2.9096 \\ 2.4006 & 2.9096 & -4.6931 \end{bmatrix}.$$

The first set of simulation results are performed to understand the role of open-loop poles and zeros on the critical value of stochastic variance,  $\sigma_*^2$ . We consider a single input single output case, where the input matrix,  $B$ , is chosen as

$$B = \begin{bmatrix} 0 & 0 & 0 & 0.0062 & 0 & 0 \end{bmatrix}^\top. \quad (34)$$

We change the output of the system as shown in Table I to determine the impact of open-loop zeros on  $\sigma_*^2$ . With single input single output, the critical value for  $\sigma_*^2$  can be computed using the  $\mathcal{H}_2$  norm of the system where the  $A$  matrix is given by Eq.

(33) and the  $B$  matrix is given in Eq. (34). The  $C$  matrix is varied as shown in Table I to determine the impact of open-loop zeros on  $\sigma_*^2$ . The matrices,  $B, C$  are chosen such that, the pairs  $(A, B)$  and  $(A, C)$  are respectively controllable and observable. The eigenvalues and, hence, the poles of the open-loop system are given as follows:

$$\lambda \in \{-1.6742, -1.0927, 0.5255, 0.2683, 0, -0.1339\}. \quad (35)$$

TABLE I: Effect of non-minimum phase zeros on critical variance ( $\sigma_*^2$ )

Output	$y = \omega_1$	$y = \omega_1 + \omega_2$	$y = \omega_2$
Critical variance	0.43	0.37	$1.6 \times 10^{-4}$
open-loop zeros	$\{-1.63, -0.8, 0.41, -0.18, 0.11\}$	$\{-1.63, -0.8, 0.4, -0.17, 0.12\}$	$\{-1.8, -0.8, 0.55, -10^{-4}\}$

The directions corresponding to the non-minimum phase zeros (unstable zeros) in state space needs more input energy to control and has less output energy to observe. In other words, the non-minimum phase zeros increase the phase lag of the system and the controller must utilize extra effort to nullify its effect. This increases the overall  $\mathcal{H}_2$  norm for the closed-loop system. Hence, the uncertainty that can be tolerated by a system with non-minimum phase zeros far away from the imaginary axis is very small [2, Chapters 5,6].

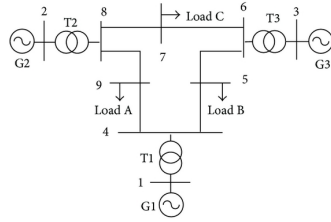


Fig. 3: WSCC 9 bus system

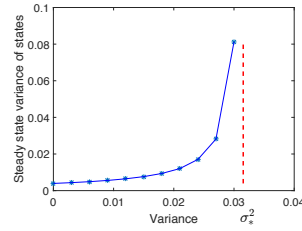


Fig. 4: Variation of steady state variance of states

Next, we consider the power network dynamics with all the inputs and outputs. As there are 3 generators, there are three inputs corresponding to the mechanical torque input of generators and three outputs corresponding to the frequency state,  $\omega$  of generators. The resultant system is controllable and observable. Assume, enters both at the input and output channels with identical variance,  $\sigma^2$ . With  $\theta$  fixed, a dynamic stabilizing feedback controller is designed by solving the LMI-based optimization problem using CVX package in MATLAB. Now, applying Theorem 11, we obtain  $\sigma_*^2 = 0.031$ . To verify the theoretical prediction for the critical value of  $\sigma_*^2$ , we compute the steady state covariance for the closed-loop system for varying values of  $\sigma^2$ . The corresponding plot is shown in Fig. 4 and observe the covariance grows unbounded as the critical value of  $\sigma_*^2$  is approached. Further, a non-robust controller based on observer feedback is designed and the critical variance of the corresponding system is  $\sigma_*^2 = 0.0093$ . This clearly demonstrates the advantage of the proposed framework.

The next set of simulation results are performed to verify the fundamental limitation results. To verify the fundamental limitation results, we assume full state feedback, i.e.,  $C = I$  and single input, where the  $B$  matrix is assumed  $B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.0802 & 0 \end{bmatrix}^T$ . The critical value for  $\sigma_*$ , following results from Theorem 15 is given by  $\sigma_* = \mu \sqrt{2 \sum_i \lambda_i(A)^{-1}}$ . Assuming the mean value of uncertainty  $\mu = 1$  and using the pole locations from Eq. (35), we obtain  $\sigma_* = 0.793$ .

## VI. CONCLUSION

Necessary and sufficient conditions for mean square exponential stability of continuous-time LTI systems with input and output channel uncertainties are derived. The mean square exponential stability results are given in terms of a spectral radius condition, which includes the computation of  $\mathcal{H}_2$  norms of the SISO deterministic systems. Further, we show the mean square exponential stability can be verified by computing the mean square system norm posed as an optimization problem using LMI's. Mean square system norm-based LMI's formulation is used for the synthesis of controller robust to input and output channel uncertainties. We also derive fundamental limitations results that arise in the mean square exponential stabilization of single input LTI system with input channel uncertainty. These results generalize existing results for discrete-time linear and nonlinear system, where the limitations are expressed in terms of the eigenvalues of open-loop system dynamics. Simulation results involving network power system are presented to demonstrate the application of the developed framework.

## REFERENCES

- [1] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory*. Springer-Verlag, New York, 1999.
- [2] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: analysis and design*. Wiley New York, 2007, vol. 2.
- [3] R. Z. Has'minskii, *Stochastic Stability of differential equations*. Germantown ,MD: Sijthoff & Noordhoff, 1980.
- [4] W. M. Wonham, "Optimal stationary control of a linear system with state-dependent noise," *SIAM Journal on Control*, vol. 5, no. 3, pp. 486–500, 1967.
- [5] J. C. Willems and G. L. Blankenship, "Frequency domain stability criteria for stochastic systems," *IEEE Transactions on Automatic Control*, vol. 16, no. 4, pp. 292–299, 1971.
- [6] P. J. McLane, "Optimal stochastic control of linear systems with state-and control-dependent disturbances," *Automatic Control, IEEE Transactions on*, vol. 16, no. 6, pp. 793–798, 1971.
- [7] J. L. Willems and J. C. Willems, "Feedback stabilizability for stochastic systems with state and control dependent noise," *Automatica*, vol. 12, no. 3, pp. 277–283, 1976.
- [8] —, "Robust stabilization of uncertain systems," *SIAM Journal on Control and Optimization*, vol. 21, no. 3, pp. 352–374, 1983.
- [9] A. El Bouhtouri and A. Pritchard, "Stability radii of linear systems with respect to stochastic perturbations," *Systems and Control Letters*, vol. 19, no. 1, pp. 29–33, 1992.
- [10] —, "A riccati equation approach to maximizing the stability radius of a linear system by state feedback under structured stochastic lipschitzian perturbations," *Systems and Control Letters*, vol. 21, pp. 475–484, 1993.
- [11] D. Bernstein, "Robust Static and Dynamic Output-Feedback Stabilization: Deterministic and Stochastic Perspectives," *IEE Trans. of Automatic Control*, vol. 32, pp. 1076–1084, 1987.
- [12] L. E. Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Systems and Control Letters*, vol. 24, pp. 223–228, 1995.
- [13] V. Dragan, A. Halanay, and A. Stoica, "A small gain theorem for linear stochastic systems," *Systems and Control Letters*, vol. 30, no. 5, pp. 243–251, 1997.
- [14] H. Deng, M. Kristić, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Transactions of Automatic Control*, vol. 46, pp. 1237–1253, 2001.
- [15] H. Deng and M. Kristić, "Output-feedback stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *Systems and Control Letters*, vol. 39, pp. 173–182, 2000.
- [16] P. Florchinger, "Lyapunov like techniques for stochastic stability," *SIAM J. Control Optim.*, vol. 33, pp. 1151–1169, 1995.
- [17] A. Diwadkar, S. Dasgupta, and U. Vaidya, "Control of systems in lure form over erasure channels," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 15, pp. 2787–2802, 2015.
- [18] L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of IEEE*, vol. 95, no. 1, pp. 163–187, 2007.
- [19] N. Elia, "Remote stabilization over fading channels," *Systems and Control Letters*, vol. 54, pp. 237–249, 2005.
- [20] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1196–1201, 2004.
- [21] N.C. Martins, M.A. Dahleh, and N.Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 438–447, 2006.
- [22] N. Elia and J. N. Eisenbeis, "Limitations of linear control over packet drop networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 826–841, 2011.

- [23] V. Gupta and B. Hassibi and R. M. Murray, "Optimal LQG Control Across Packet-Dropping Links," *System and Control Letters*, vol. 56, no. 6, pp. 439–446, 2007.
- [24] A. Diwadkar and U. Vaidya, "Limitations for nonlinear observation over erasure channel," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 454–459, 2013.
- [25] U. Vaidya and N. Elia, "Limitations of nonlinear stabilization over erasure channels," in *2010 49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010, pp. 7551–7556.
- [26] A. Diwadkar and U. Vaidya, "Robust synchronization in nonlinear network with link failure uncertainty," in *2011 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*. IEEE, 2011, pp. 6325–6330.
- [27] U. Vaidya and N. Elia, "Limitation on nonlinear stabilization over packet-drop channels: Scalar case," *Systems and Control Letters*, vol. 61, no. 9, pp. 959–966, 2012.
- [28] A. Diwadkar, S. Dasgupta, and U. Vaidya, "Stochastic positive real lemma and synchronization over uncertain network," in *2014 IEEE 53rd Annual Conference on Decision and Control (CDC)*. IEEE, 2014, pp. 4560–4565.
- [29] A. Diwadkar and U. Vaidya, "Limitations and tradeoffs in synchronization of large-scale networks with uncertain links," *Scientific reports*, vol. 6, pp. 1–11, 2016.
- [30] N. Elia, J. Wang, and X. Ma, *Mean Square Limitations of Spatially Invariant Networked Systems*. Control of Cyber-Physical Systems: Lecture Notes in Control and Information Sciences, 2013, vol. 499, pp 357-378.
- [31] A. Diwadkar and U. Vaidya, "Stabilization of linear time varying systems over uncertain channels," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 7, pp. 1205–1220, 2014.
- [32] J. Lu and R. E. Skelton, "Mean-square small gain theorem for stochastic control: Discrete-time case," *IEEE Transactions of Automatic Control*, vol. 47, no. 3, pp. 490–494, 2002.
- [33] B. Bamieh, "Structured stochastic uncertainty," in *Communication, Control, and Computing (Allerton), 2012 50th Annual Allerton Conference on*. IEEE, 2012, pp. 1498–1503.
- [34] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*. New York: Springer-Verlag, 1994.
- [35] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-time Markov jump linear systems*. Springer Science & Business Media, 2006.
- [36] S. Pushpak, A. Diwadkar, and U. Vaidya, "Stability analysis and controller synthesis for continuous-time linear stochastic systems," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 3792–3797.
- [37] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, 1997.
- [38] R. A. Horn and C. J. Johnson, *Matrix Analysis*. Cambridge: Cambridge University Press, 1985.
- [39] M. A. Rotea, "The generalized  $H_2$  control problem," *Automatica*, vol. 29, no. 2, pp. 373–385, 1993.
- [40] A. AL-Hinai, "Wsc 9-bus system," 2000. [Online]. Available: <http://publish.illinois.edu/smartergrid/wsc-9-bus-system/>